Understanding the quantum world with a tennis racket: How classical mechanics helps control qubits

QuSCo Seminar

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Collaboration and Fundings

A joint work between mathematicians, physicists and chemists

- Group of S. J. Glaser (Munich, Germany)
- Group of P. Mardesic (Dijon, France)

IAS TUM Institute for Advanced Study

Deutsche Forschungsgemeinschaft DFG

EIPHI-BFC GRADUATE SCHOOL CROSS DISCIPLINARY SCIENCE AND TECHNOLOGY

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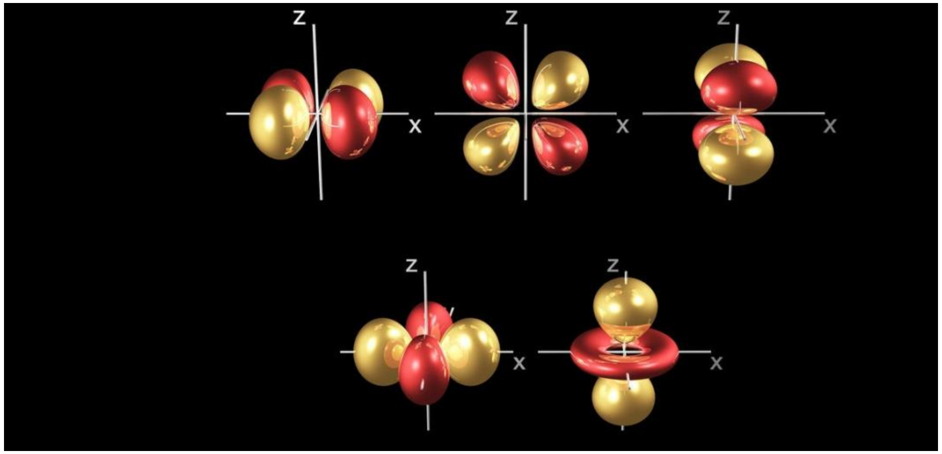
Introduction to Quantum control

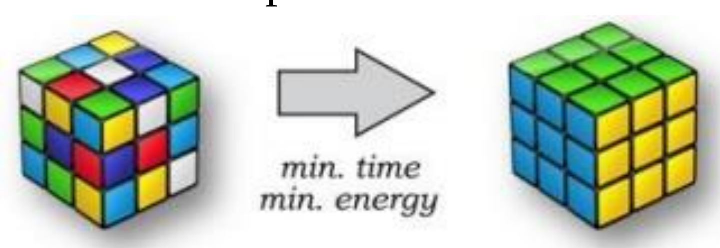


Quantum effects:
Atomic orbitals (Probability of 95% to find the electron)

Quantum theory:
Theoretical basis of modern physics that explains the nature and behavior of matter and energy at the atomic level.

Fundamental quantum effects → Control theory → Quantum technologies

Control theory: Realization of basic operations



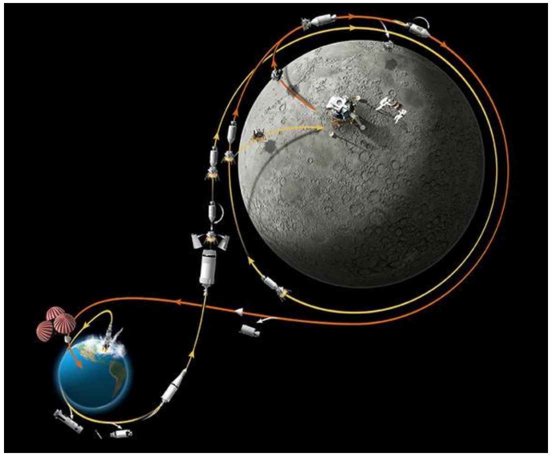
Quantum control

Manipulating the quantum dynamics of atoms, molecules and spins with external electromagnetic fields.

Design of specific electric or magnetic fields

Application of tools of control theory (Optimal control theory) to quantum physics

A famous example in classical physics:



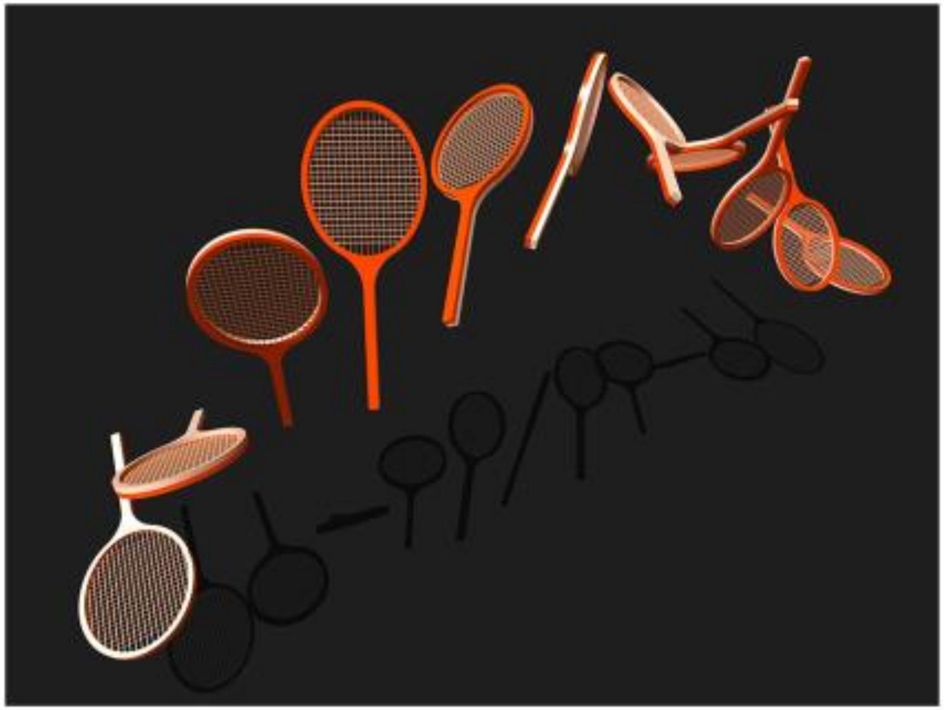
Apollo and Smart I

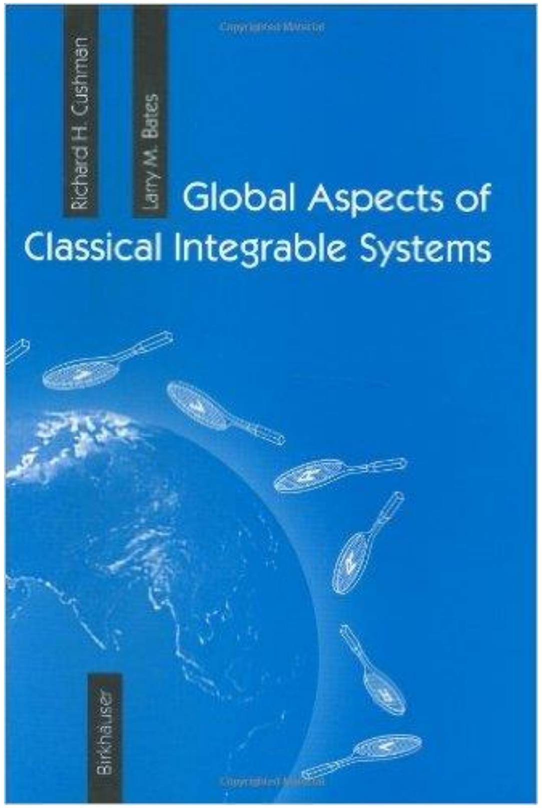
How do we build up physical intuition in quantum control ?

Analogy with classical physics

The Tennis Racket Effect

The tennis racket effect

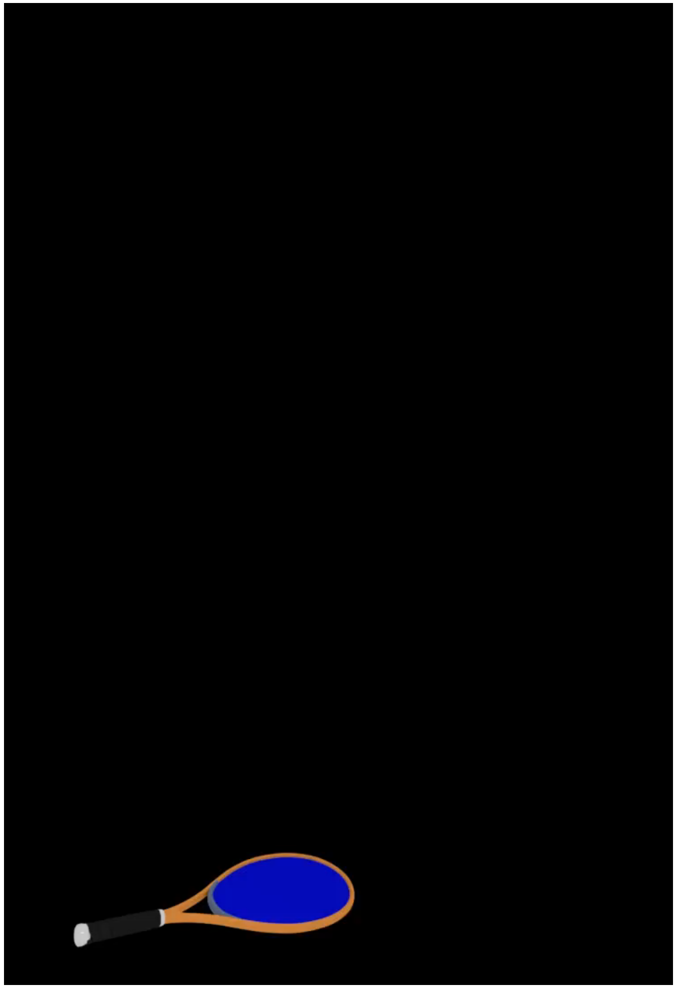




R. H. Cushman and L. M. Bates

Geometric effect that can be observed in any three-dimensional asymmetric rigid body.

The Tennis Racket Effect



How to control a skate board with the tennis racket effect

According to the tennis racket effect, the Monster Flip is impossible.

It can be shown that it is possible, but with a very low probability….

References about the tennis racket effect

Scientific papers:

- M. S. Ashbaugh, C. C. Chiconc and R. H. Cushman, The Twisting Tennis Racket, J. Dyn. Diff. Eq. 3, 67 (1991).
- R. H. Cushman and L. Bates, Global Aspects of Classical Integrable Systems (Birkhauser, Basel, 1997).
- L. Van Damme, P. Mardesic and D. Sugny, The tennis racket effect in a three dimensional rigid body, Physica D 338, 17 (2017)
- P. Mardesic, G. J. Gutierrez Guillen, L. Van Damme and D. Sugny, Phys. Rev. Lett. 125, 064301 (2020)

Popular studies:

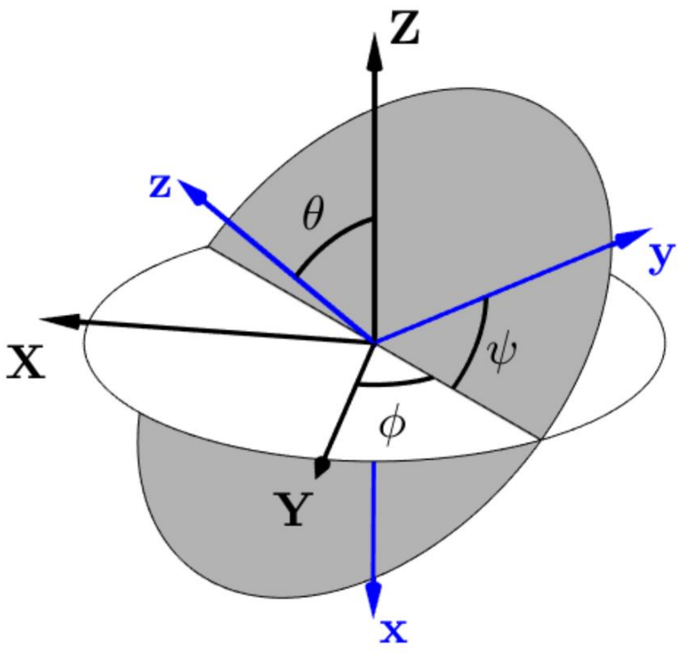
- Images des Mathématiques (Mardesic and Sugny, 2019)
- Le monde (D. Larousserie, Mardesic and Sugny, 2020)
- Movies on Youtube: Physics girls (2019), the Monster Flip….
- The Dzhanibekov effect, Wikipedia page…

Classical dynamics of a three-dimensional rigid body

The rotational dynamics of a three-dimensional rigid body is described by an integrable Hamiltonian system.

The position of the rigid body is given by an element of SO(3).

The three Euler angles are used as coordinates.



Two frames:

- A space-fixed frame (X,Y,Z)
- A body-fixed frame (x,y,z)

Ref.: V. I. Arnold, Mathematical methods of Classical Mechanics

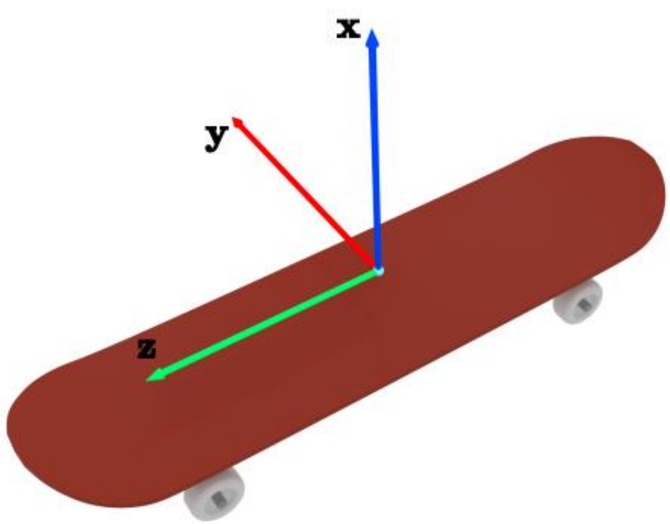
Axes and moments of inertia

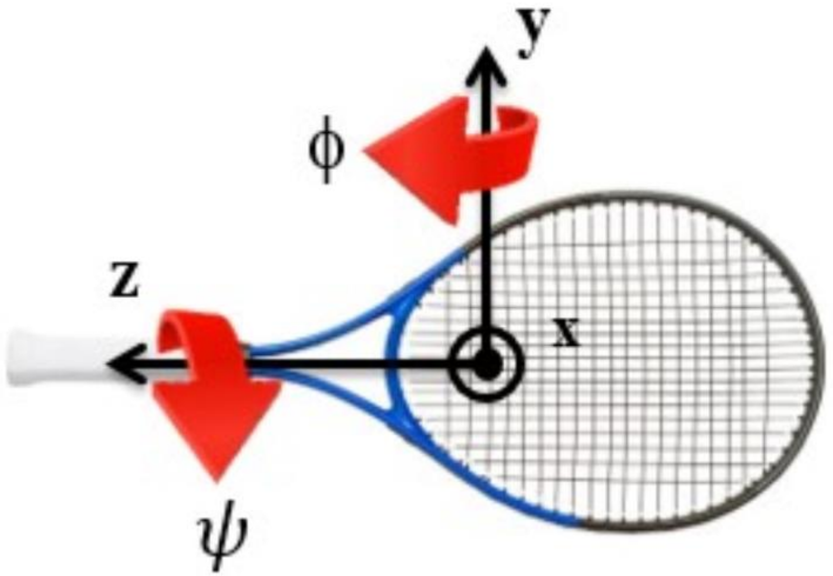
Mass repartition: Inertia matrix

- ➢ Eigenvectors: Inertia axes
- ➢ Eigenvalues: Inertia moments

$$\boxed{I_{jk} = \int_V \rho(\mathbf{r})(r^2\delta_{jk} - x_j x_k)d^3\mathbf{r},}$$

Convention: $\boxed{I_z < I_y < I_x}$





Classical dynamics of a three-dimensional rigid body

The dimension of the phase space is 6.

In the absence of outside forces, there are four first integrals (the angular momentum M and the energy): The Euler top.

For a regular point, the dynamics are restricted to a two-dimensional torus.

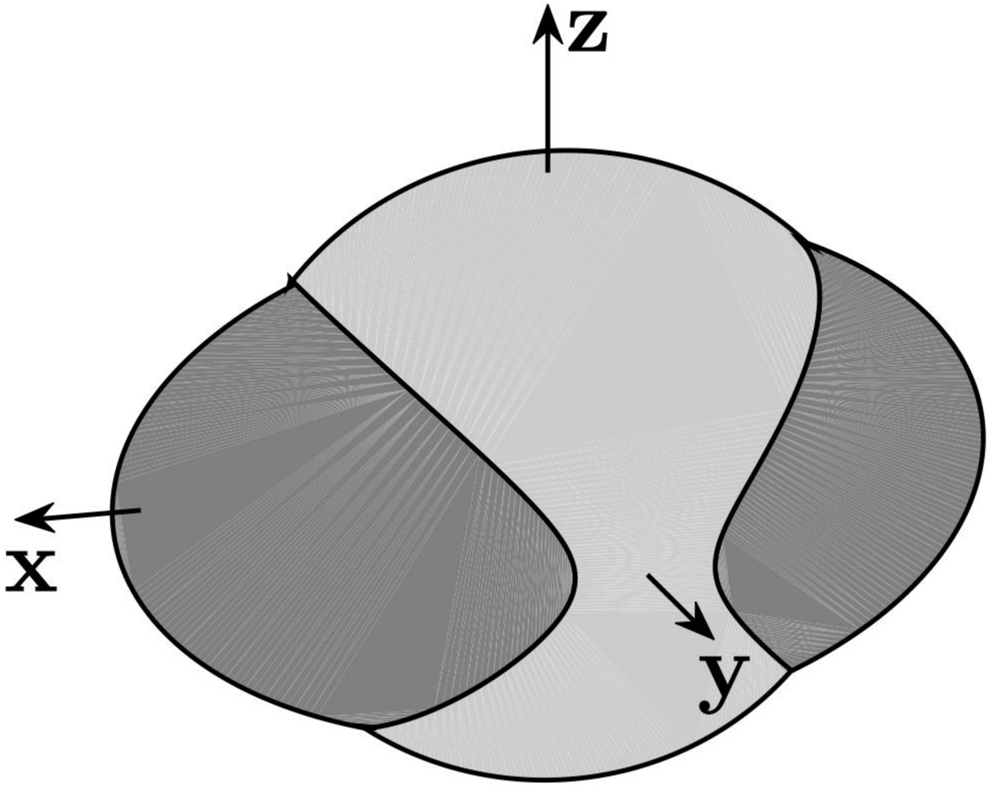
In the reduced phase space (M_x,M_y,M_z), the trajectory is the intersection of two surfaces:

$$\boxed{\begin{cases} 2E = \dfrac{M_x^2}{I_x} + \dfrac{M_y^2}{I_y} + \dfrac{M_z^2}{I_z} \\ M^2 = M_x^2 + M_y^2 + M_z^2 \end{cases}}$$

Rem.: Extension to a n-dimensional rigid body with the Lax pair approach

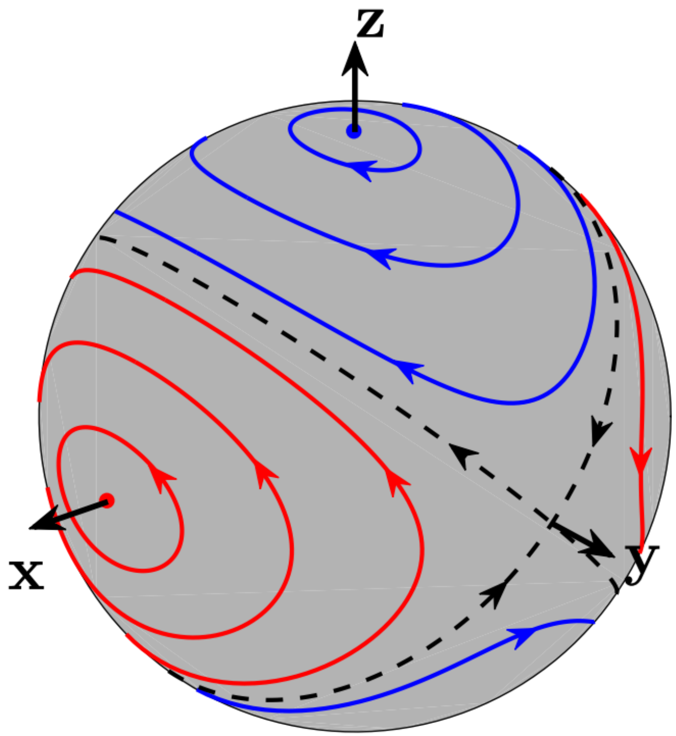
Classical dynamics of a three-dimensional rigid body

$$\begin{cases} 2E = \dfrac{M_x^2}{I_x} + \dfrac{M_y^2}{I_y} + \dfrac{M_z^2}{I_z} \\ M^2 = M_x^2 + M_y^2 + M_z^2 \end{cases}$$



Intersection of a sphere and an ellipsoid.

Classical dynamics of a three-dimensional rigid body

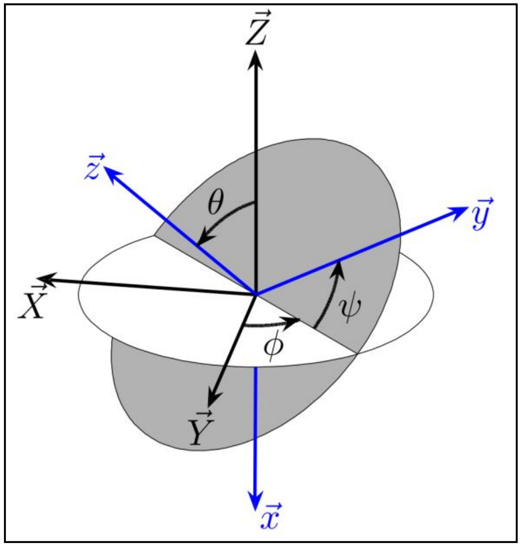


Reduced phase space:

- Rotating and oscillating trajectories, separatrix
- Four stable and two unstable equilibrium points.

Mathematical description of the tennis racket effect

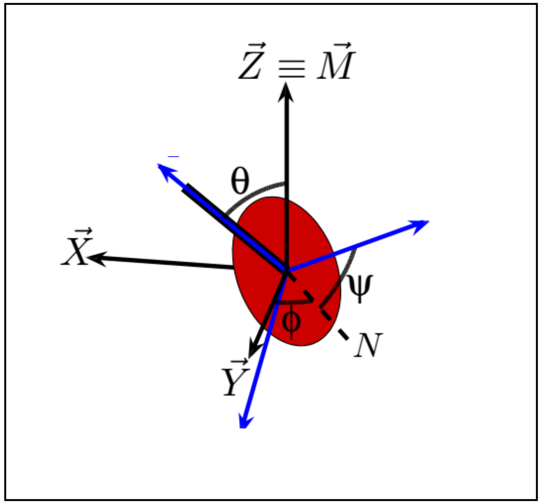
Definition of a particular set of Euler angles:



Tennis racket effect :

$$\boxed{\Delta\phi = 2\pi, \quad \Delta\psi \sim \pi}$$

$$\boxed{\theta \approx \frac{\pi}{2}}$$



Mathematical description of the tennis racket effect

Angular momentum: Rotational equivalent of the momentum

$$\boxed{M_i = I_i \omega_i}$$

Angular momentum

Angular velocity

Euler's equations: Dynamics of the angular momentum in the frame attached to the racket

$$\left\{\begin{aligned} \dot{M}_x &= -\left(\frac{1}{I_y} - \frac{1}{I_z}\right) M_y M_z \\ \dot{M}_y &= \left(\frac{1}{I_x} - \frac{1}{I_z}\right) M_x M_z \\ \dot{M}_z &= -\left(\frac{1}{I_x} - \frac{1}{I_y}\right) M_x M_y \end{aligned}\right.$$

Constants of the motion:

$$\left\{\begin{aligned} E &= \frac{M_x^2}{2I_x} + \frac{M_y^2}{2I_y} + \frac{M_z^2}{2I_z} \\ M^2 &= M_x^2 + M_y^2 + M_z^2 \end{aligned}\right.$$

Integrable system (Euler top)

Mathematical description of the tennis racket effect

Euler's equations: Dynamics of the Euler angles

$$\begin{cases} M_x = -M\sin\theta\cos\psi \\ M_y = M\sin\theta\sin\psi \\ M_z = M\cos\theta \end{cases}$$

the two angles described the dynamics in the reduced phase space.

The dynamics of the third angle is given by the angular velocity.

$$\begin{cases} \dot{\theta} = M\left(\frac{1}{I_y} - \frac{1}{I_x}\right)\sin\theta\sin\psi\cos\psi \\ \dot{\phi} = M\left(\frac{\sin^2\psi}{I_y} + \frac{\cos^2\psi}{I_x}\right) \\ \dot{\psi} = M\left(\frac{1}{I_z} - \frac{\sin^2\psi}{I_y} + \frac{\cos^2\psi}{I_x}\right)\cos\theta \end{cases}$$

We introduce the following coefficients

$$\begin{cases} a = \frac{I_y}{I_z} - 1 \\ b = 1 - \frac{I_y}{I_x} \\ c = \frac{2I_y E}{M^2} - 1 \end{cases}$$

Perfect asymmetric rigid body: $ab \rightarrow +\infty$

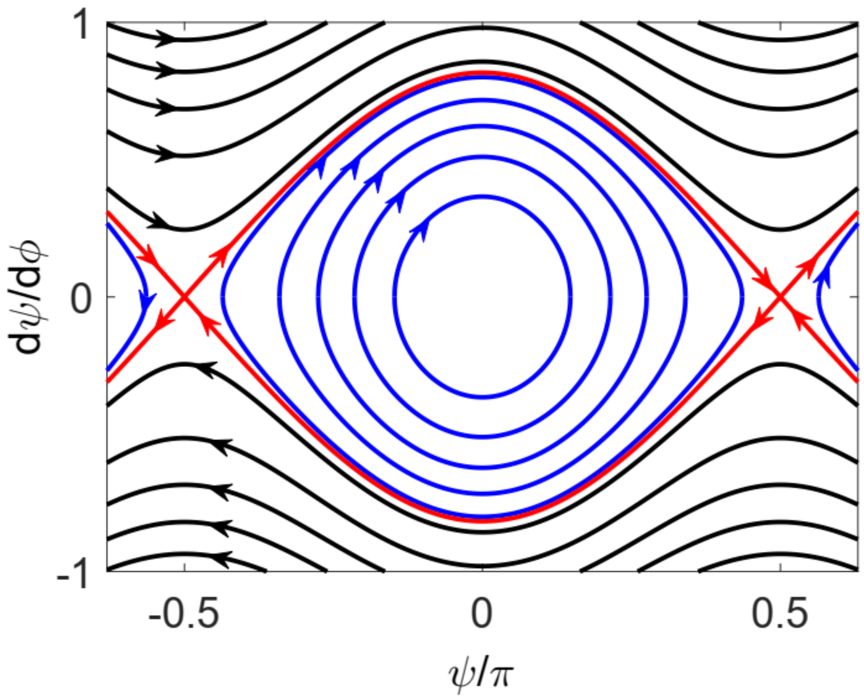
Mathematical description of the tennis racket effect

The tennis racket effect is a geometric effect which does not depend directly on the duration of the process.

We can reduce the dynamics to consider only two angles: $\boxed{\dot{\psi} = \frac{d\psi}{d\phi}}$

$$\boxed{\frac{d\psi}{d\phi} = \pm \frac{\sqrt{(a + b\cos^2\psi)(c + b\cos^2\psi)}}{1 - b\cos^2\psi}}$$

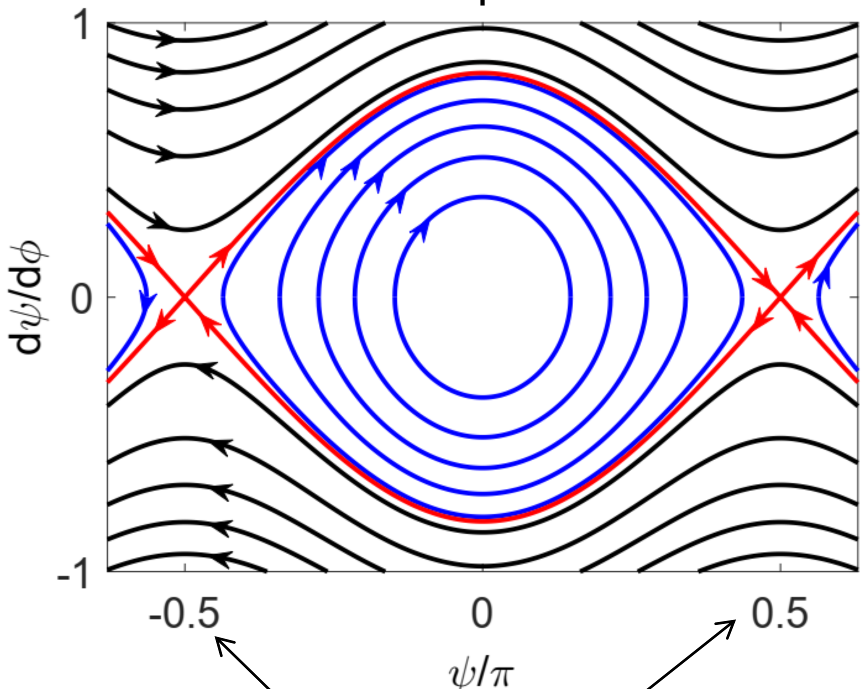
Phase space



Mathematical description of the tennis racket effect

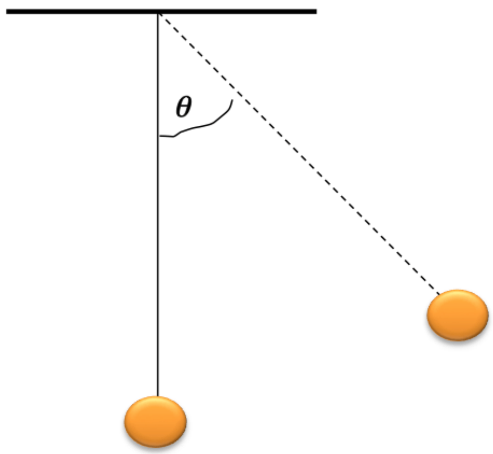
Analogy with a standard planar pendulum:

Phase space

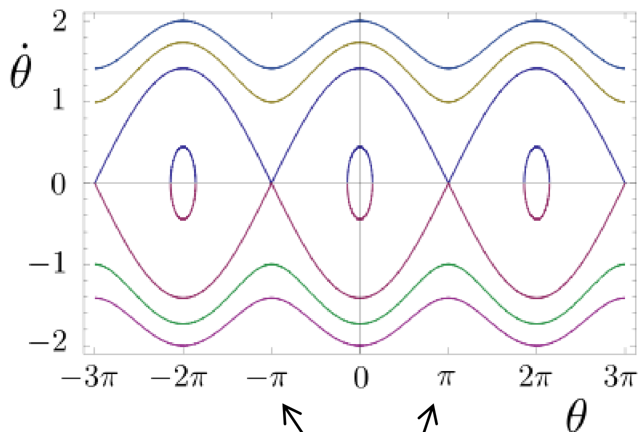


A variation of π

Tennis racket effect



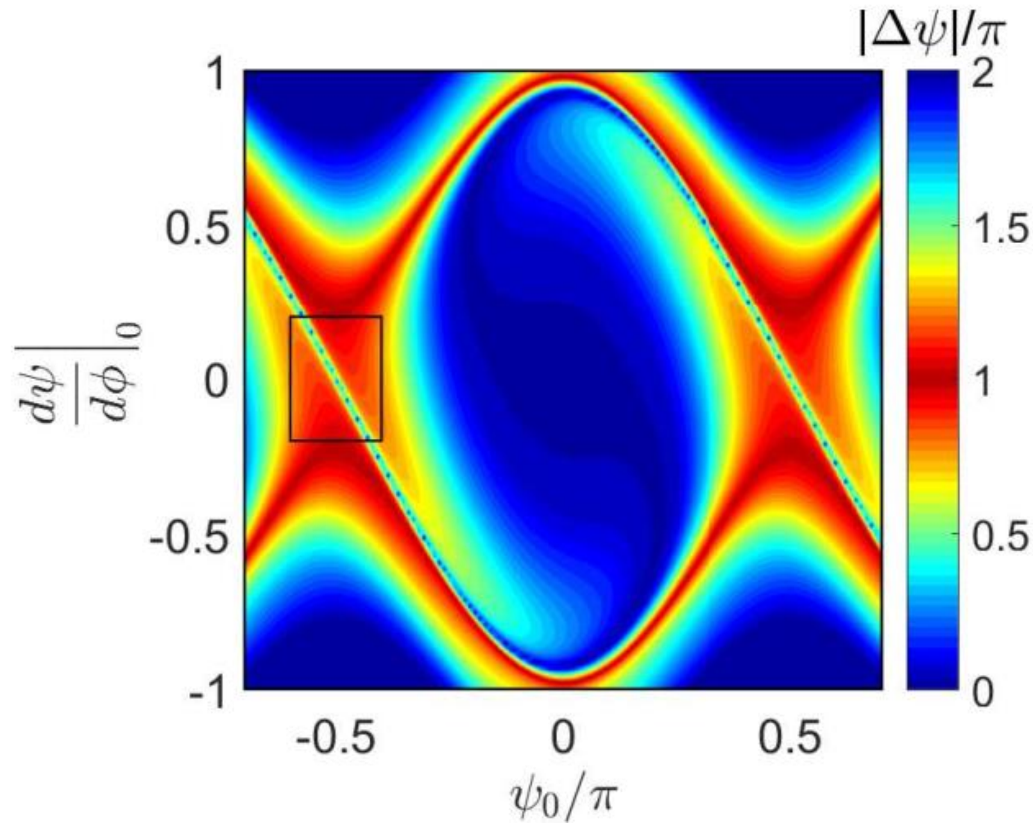
Phase space



A variation of 2π

Mathematical description of the tennis racket effect

Robustness of the tennis racket effect against initial conditions:



- What is the geometric origin of the tennis racket effect ?
- Is it possible to estimate the robustness of the effect ?
- 3 parameters (a,b,c)

Mathematical description of the tennis racket effect

We consider a symmetric configuration: $\boxed{\psi_0 = -\frac{\pi}{2} + \varepsilon \rightarrow \psi_f = \frac{\pi}{2} - \varepsilon}$

The new parameter is the defect to a perfect tennis racket effect.

Using a change of variable and the parity of the integral:

$$\boxed{\Delta\phi(\varepsilon) = \int_{\sin^2\varepsilon}^{1} \frac{1}{b} \frac{(1-bx)dx}{\sqrt{x(x-\beta)(1-x)(x-\alpha)}}}$$

$$x = \cos^2\psi; \alpha = -\frac{a}{b}; \beta = -\frac{c}{b}$$

Incomplete elliptic integral depending on the different parameters of the problem.

We study the solution of the following equation:

$$\boxed{\Delta\phi_{a,b,c}(\varepsilon) = 2\pi}$$

Mathematical description of the tennis racket effect

$$\boxed{\Delta\phi(\varepsilon) = \int_{\sin^2\varepsilon}^{1} \frac{1}{b} \frac{(1-bx)dx}{\sqrt{x(x-\beta)(1-x)(x-\alpha)}}}$$

We complexify the *x*- coordinate and we introduce a Riemann surface:

$$\boxed{y^2 = x(x-\beta)(1-x)(x-\alpha)}$$

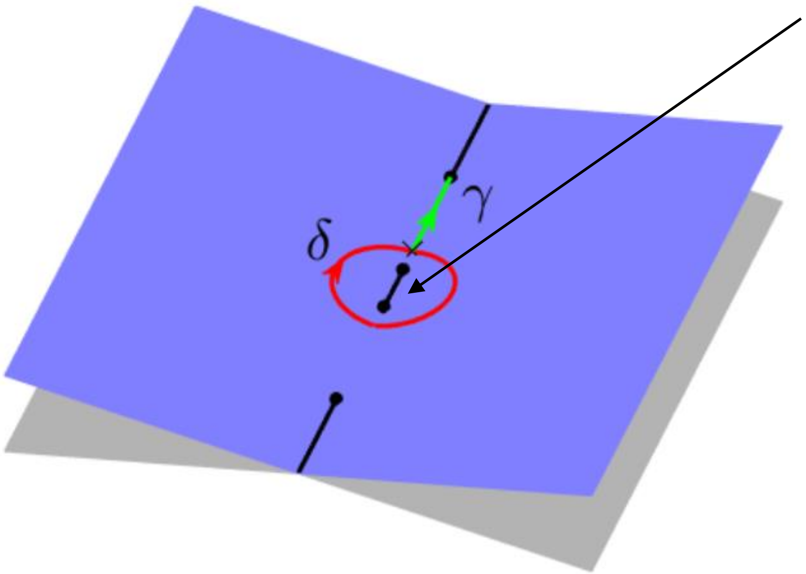
The integral is interpreted as an Abelian integral over this surface.

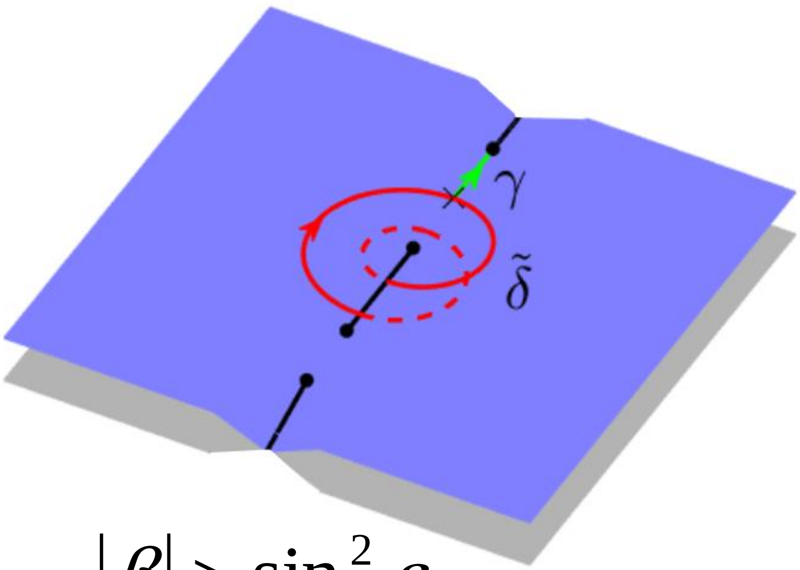
Its behavior is given by the geometry and the **singularity** of the surface.

Mathematical description of the tennis racket effect

Two different configurations:

A pole appears when c goes to 0





$$|\beta| < \sin^2 \varepsilon$$

$$\gamma \to \gamma + \delta; \int_{\delta} \omega \neq 0$$

$$|\beta| > \sin^2 \varepsilon$$

$$\gamma \to \gamma + \tilde{\delta}; \int_{\tilde{\delta}} \omega = 0$$

In the first case, by the Picard-Lefschetz formula, the integration contour is deformed to itself plus a loop around the singularity.

This property reveals the multi-valued character of the function: a logarithmic function. No logarithmic divergence in the second case !

Mathematical description of the tennis racket effect

We deduce:

$$\boxed{\Delta\phi(\varepsilon) = \frac{1}{\sqrt{ab}} h_{a,b,c}(\sin^2\varepsilon) - \frac{1}{\sqrt{ab}} \ln(\sin^2\varepsilon)}$$

Bounded and analytic function (m is given by the bound of *h*).

Theorem of the Tennis Racket Effect:

For all c such that: $|c| < b\exp(-2\pi\sqrt{ab} - m)$

For ab large enough, the equation $\Delta\phi_{a,b,c}(\varepsilon) = 2\pi$

has a unique solution which verifies:

$$\arcsin\left(\sqrt{\left|\frac{c}{b}\right|}\right) < \varepsilon_S < \arcsin\left(\exp\left(-\pi\sqrt{ab} - \frac{m}{2}\right)\right)$$

This leads to:

$$\lim_{ab\to+\infty} \varepsilon_S(a,b,c) = 0$$

Mathematical description of the tennis racket effect

Estimation of the robustness of the tennis racket effect:

$$\Delta\varphi = 2\pi \quad \Delta\psi = \pi - \varepsilon$$

$$\boxed{\varepsilon \approx \exp(-\sqrt{ab}\,\frac{\Delta\phi}{2})}$$

Robustness with respect to the shape of the body

$$\boxed{a = \frac{I_y}{I_z} - 1; b = 1 - \frac{I_y}{I_x}}$$

Refs.: L. Van Damme et al, Physica D (2017)

The Monster Flip effect

The same analysis can be conducted for the Monster Flip effect.

$$\boxed{\Delta\phi = 2\int_{\psi_i}^{\pi/2+\varepsilon} \frac{1-b\cos^2\psi}{\sqrt{(a+b\cos^2\psi)(c+b\cos^2\psi)}}\,d\psi}$$

We arrive at:

$$\boxed{\Delta\phi(\varepsilon) = \frac{1}{\sqrt{ab}}h_{a,b,c}(\sin^2\varepsilon) + \frac{1}{\sqrt{ab}}\ln(\sin^2\varepsilon)}$$

$$\boxed{\varepsilon \approx \frac{\sqrt{|\beta|}}{2}\exp(\pi\sqrt{ab})}$$

This parameter has to be very small

How to use this effect in the quantum world ?

Formal equivalence between the Euler equations and the Bloch equations:

$$\dot{\vec{M}} = \begin{pmatrix} 0 & -M_z/I_z & M_y/I_y \\ M_z/I_z & 0 & M_x/I_x \\ -M_y/I_y & -M_x/I_x & 0 \end{pmatrix} \vec{M} \quad \Longleftrightarrow \quad \dot{\vec{M}} = \begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix} \vec{M}$$

Euler equations

Bloch equations (spin ½, magnetic resonance)

$\vec{\mathrm{M}} \longrightarrow$ Quantum state

$\Omega_i \longrightarrow$ External control fields

Identification: $\Omega_i = \dfrac{M_i}{I_i}$ $\Rightarrow$ The moments of inertia are free parameters

How to translate this property into the quantum world ?

Formal equivalence between the Euler equations and the Bloch equations:

$$\dot{\vec{M}} = \begin{pmatrix} 0 & -M_3/I_3 & M_2/I_2 \\ M_3/I_3 & 0 & M_1/I_1 \\ -M_2/I_2 & -M_1/I_1 & 0 \end{pmatrix} \vec{M} \quad \Longleftrightarrow \quad \dot{\vec{M}} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \vec{M}$$

Euler equations Bloch equations

Identification: Specific choice of the control fields (only two fields are available)

Case (a): $\boxed{\begin{cases} \Omega_1 = M_1/I_1 = \Omega \\ I_2 = +\infty \\ \Omega_3 = M_3/I_3 = \Delta \end{cases}}$

Case (b): $\boxed{\begin{cases} \Omega_1 = M_1/I_1 \\ \Omega_2 = M_2/I_2 \\ I_3 = +\infty \end{cases}}$

Geometric control of population transfer

We consider the case (a) to illustrate the properties of the control fields.

Without loss of generality, we can set:

$$\boxed{\begin{cases} I_1 = 1 \\ I_3 = \dfrac{1}{k^2}, k \in [0,1] \end{cases}}$$

Some standard solutions of the Bloch equation can be recovered from limiting cases of the tennis racket effect:

$k \to 0:$ Pi-pulse

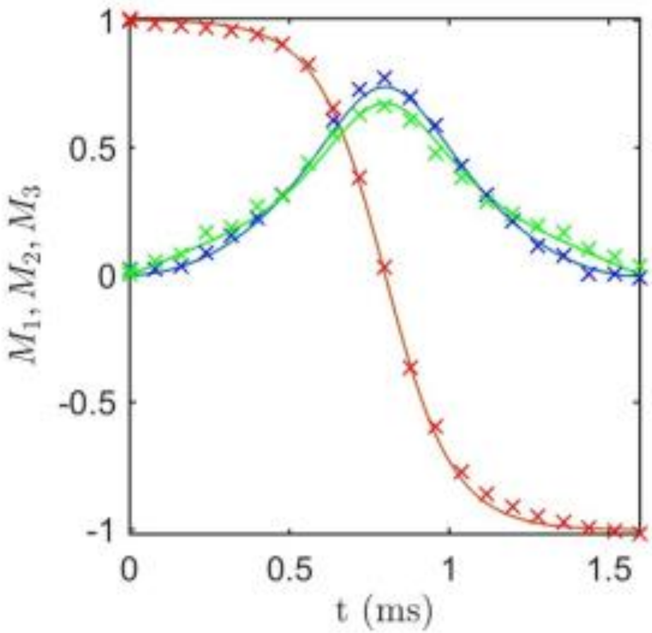
$k \to 1:$ Adiabatic pulse

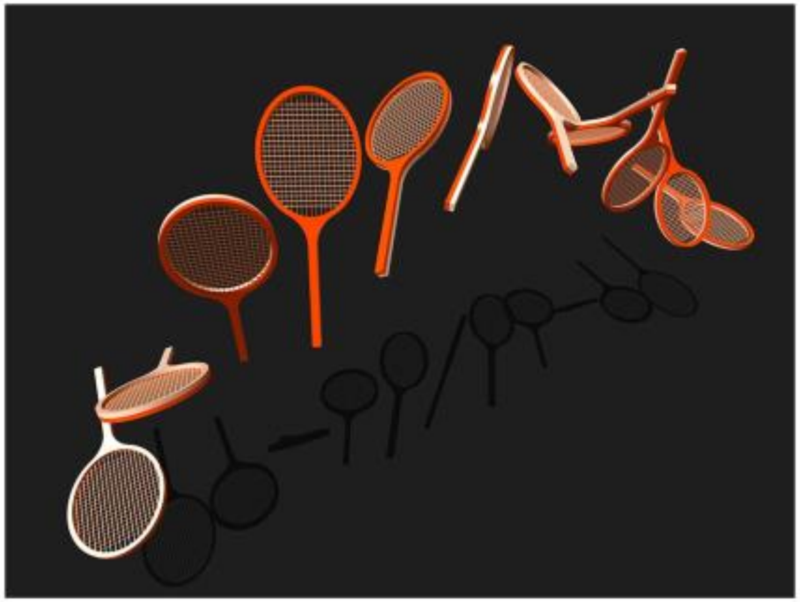
$Separatrix:$ Allen-Eberly solutions

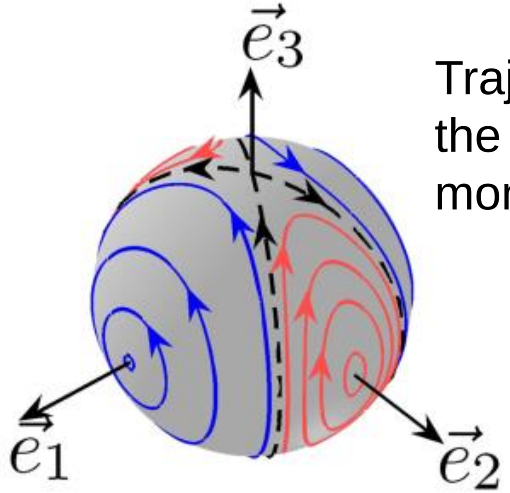
$$\boxed{\begin{aligned} \Omega &= \frac{\pm 1}{\tau\sqrt{1-k^2}} \operatorname{sec}h\left(\frac{t}{\tau}+\rho\right) \\ \Delta &= \frac{\pm k}{\tau\sqrt{1-k^2}} \tanh\left(\frac{t}{\tau}+\rho\right) \end{aligned}}$$

How to use this effect in the quantum world ?

A tennis racket effect for a spin ½ particle:







Trajectories of the angular momentum

A trajectory close to the separatrix

A robust transfer of state for the qubit

Refs.: L. Van Damme et al, Sci. Rep. (2017)

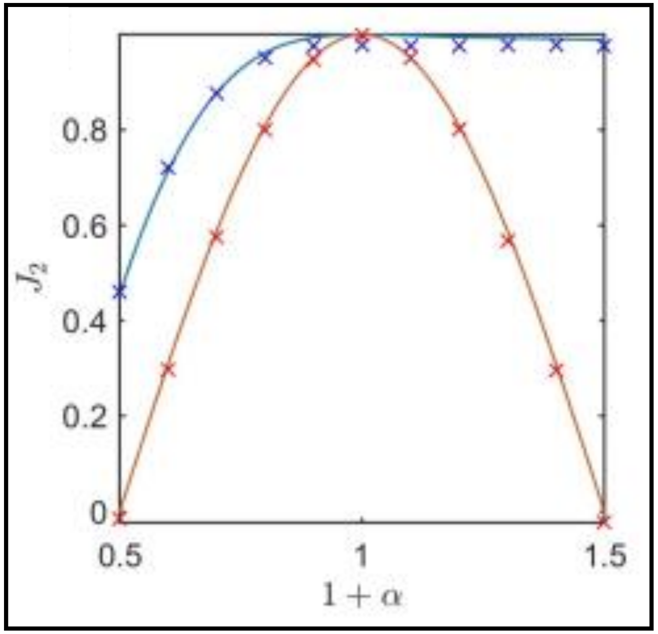
Robustness of the control process

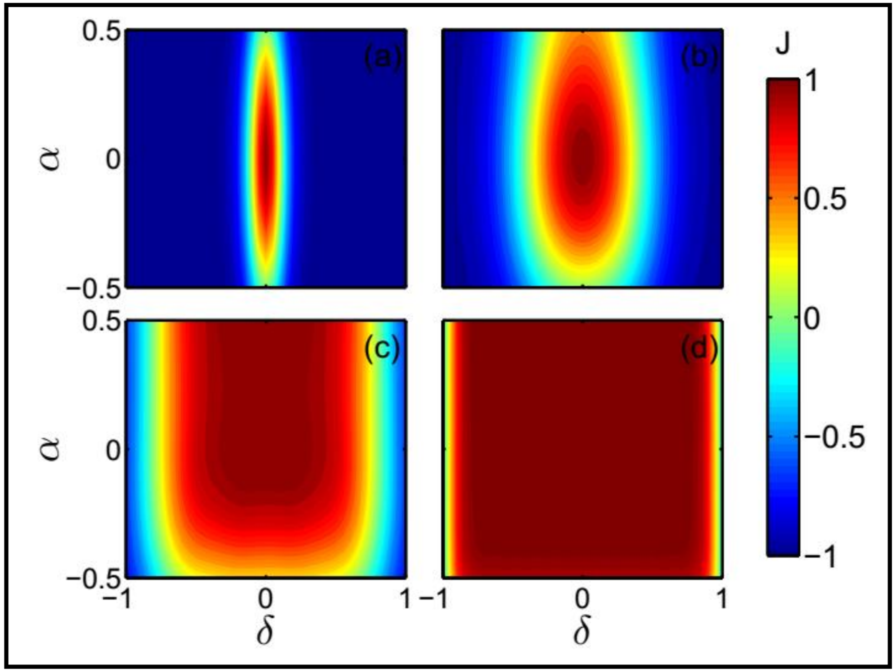
Evaluation of the robustness in the spin case:

$$\begin{cases} \Omega_{1,2}^{(\alpha)} = (1+\alpha)\Omega_{1,2} \\ \Omega_3 = \Omega_3 + \delta \end{cases}$$

$$I_x = 1; I_y = \frac{1}{k^2}; I_z = +\infty$$

$$k = 0.2; 0.6; 0.9; 0.99$$





The robustness of the process can be adjusted by choosing appropriate moments of inertia (parameter k)

Implementation of one-qubit gates

Using the Tennis Racket Effect, novel control strategies in quantum computing can be found: One-qubit gate.

Quantum phase gate: $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$

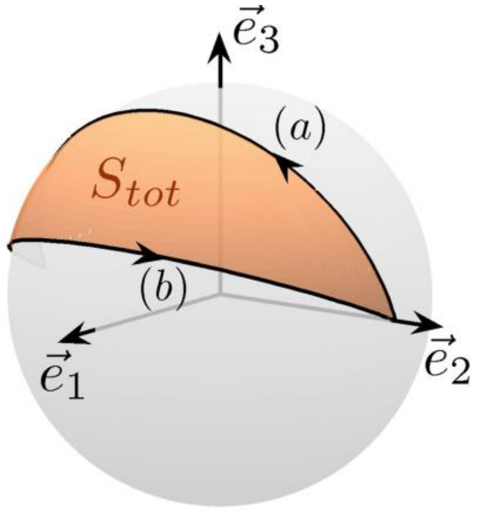
Montgomery phase: $\boxed{\Delta\varphi = \frac{2ET}{M} - S}$

Dynamical contribution　　　Geometric contribution

The dynamical contribution is not robust.

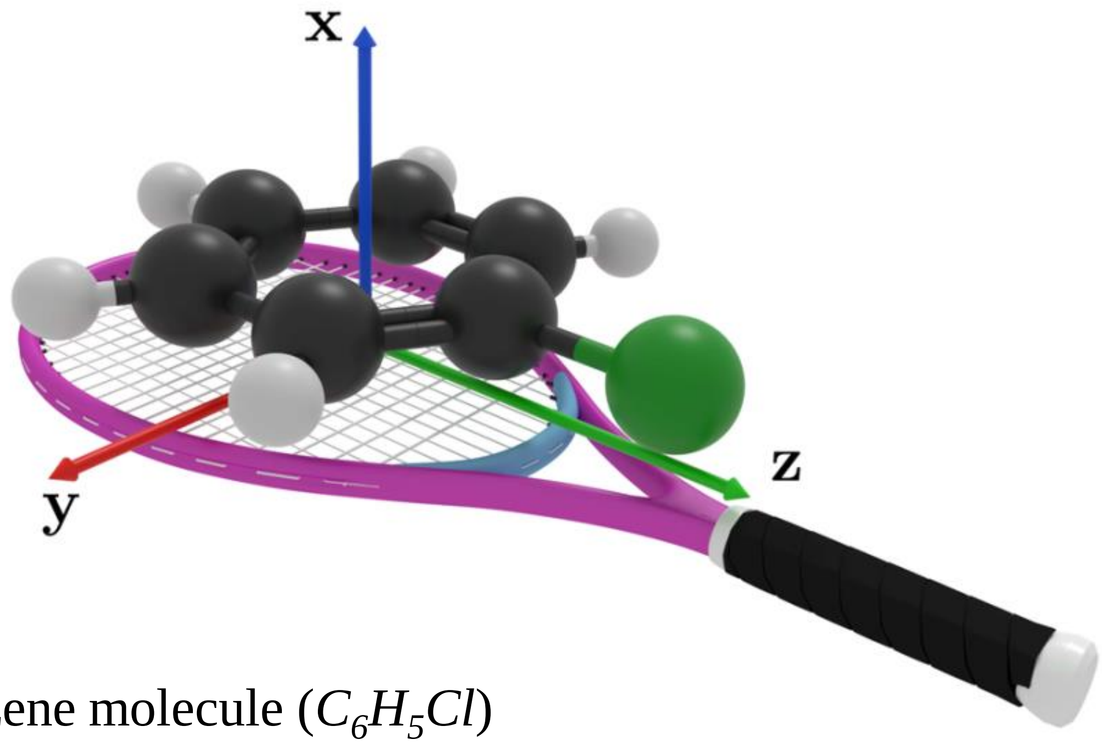
A concatenation of pulses to eliminate this contribution.

$$\boxed{\Delta\varphi = 2[\arcsin(\sqrt{1-k_a^2}) - \arcsin(\sqrt{1-k_b^2})]}$$



How to use this effect in the quantum world ?

Another idea is to consider the dynamics of asymmetric top molecules.



Chlorobenzene molecule (C_6H_5Cl)

Signature of this classical effect on:

- Spectrum of the asymmetric molecule
- Dynamics of the wave function

Conclusion and perspectives

Signature of the tennis racket effect on the wave function dynamics.

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Quantum Persistent Tennis Racket Dynamics of Nanorotors

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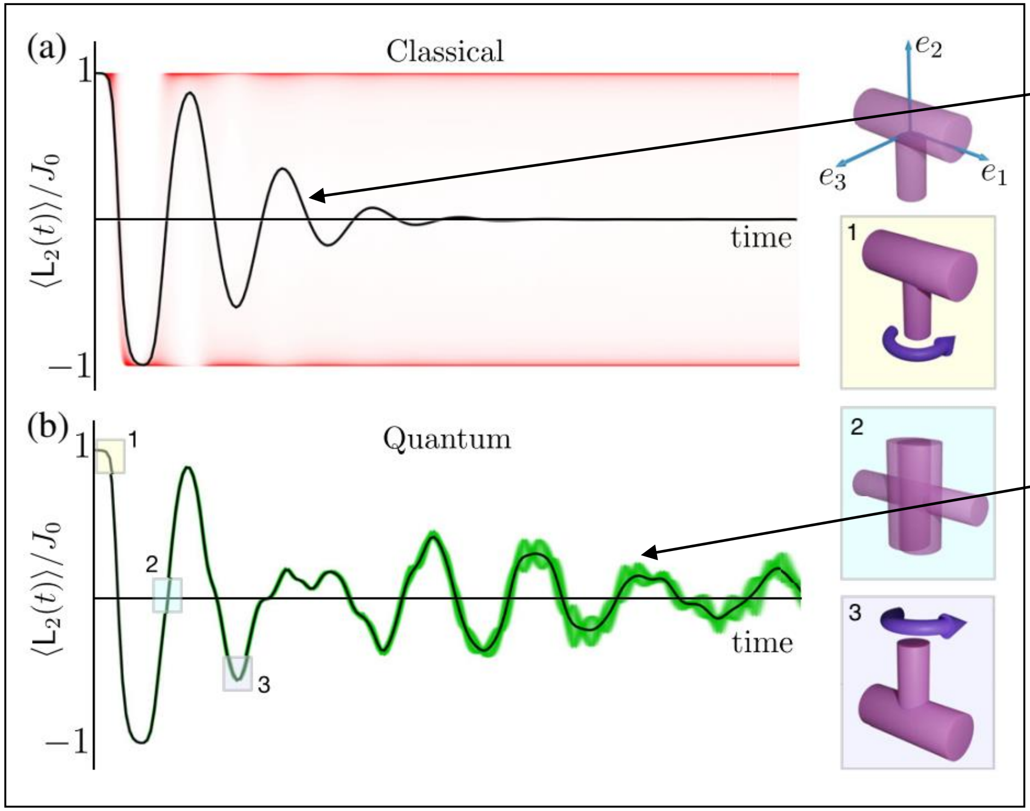

Thermally averaged trajectories

Quantum tunneling

Rem.: This effect could be observed in a neighborhood of any singular torus.

Conclusion and perspectives

Different perspectives from these results:

- A Lax pair approach of the Tennis Racket Effect (independent of the angular coordinates): Extension to SO(n)

- A rigorous semi-classical analysis of the Tennis Racket Effect in the quantum regime (singular Bohr-Sommerfeld rules)

Semi-classical limit / asymmetric limit

- Other physical or chemical applications of the Tennis Racket Effect.
- Experimental demonstrations of the quantum effect.